MATH 303 — Measures and Integration
Lecture Notes, Fall 2024

Ethan Ackelsberg

Contents

Chapter 1

Motivating Problems of Measure Theory

1.1 The Problem of Measurement

A basic (and very old) problem in mathematics is to compute the size (length, area, volume) of geometric objects. Areas of polygons and circles can be computed by elementary methods. More complicated regions bounded by continuous curves can be attacked with methods from calculus. But what about more general subsets of Euclidean space? Does it always make sense to talk about the (hyper-)volume of a subset of $\mathbb{R}^d$? What properties does volume have, and how do we compute it?

We will consider these general questions as the "problem of measurement" in Euclidean space and discuss some approaches to a solution.

1.2 Riemann Integration and Jordan Content

A good first attempt at solving the problem of measurement comes from the Riemann theory of integration. The basic strategy is to approximate general regions by finite collections of boxes (sets of the form $B = \prod_{i=1}^{d} [a_i, b_i]$). For such a box B, we declare the volume to be $\mathrm{Vol}(B) = \prod_{i=1}^{d} (b_i - a_i)$ and use this to define the volume of more general regions. We will now make this idea rigorous.

Definition 1.1. Let $B = \prod_{i=1}^{d} [a_i, b_i]$ be a box in $\mathbb{R}^d$, and let $f : B \to \mathbb{R}$ be a bounded function.

- A *Darboux partition* of B is a family of finite sequences $(x_{i,j})_{1 \le i \le d, 0 \le j \le n_i}$ such that $a_i = x_{i,0} < x_{i,1} < \cdots < x_{i,n_i} = b_i$ for each $i \in \{1, \ldots, d\}$.

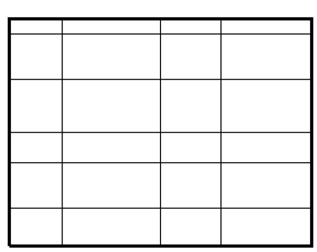

Figure 1.1: A Darboux partition in dimension $d = 2$ with $n_1 = 4$ and $n_2 = 6$.

- Given a Darboux partition $P = (x_{i,j})_{1\le i\le d, 0\le j\le n_i}$ of B, the *upper* and *lower Darboux sums of f over B* are given by

$$U_B(f,P) = \sum_{\mathbf{j}\in\prod_{i=1}^d\{1,\dots,n_i\}} \sup_{\boldsymbol{x}\in B_{\mathbf{j}}} f(\boldsymbol{x})\cdot \mathrm{Vol}(B_{\mathbf{j}})$$

and

$$L_B(f,P) = \sum_{\mathbf{j}\in\prod_{i=1}^d\{1,\dots,n_i\}} \inf_{\boldsymbol{x}\in B_{\mathbf{j}}} f(\boldsymbol{x})\cdot \mathrm{Vol}(B_{\mathbf{j}}),$$

where $B_{\mathbf{j}}$ is the box $\prod_{i=1}^d [x_{i,j_i-1}, x_{i,j_i}]$, and $\mathrm{Vol}(B_{\mathbf{j}}) = \prod_{i=1}^d (x_{i,j_i} - x_{i,j_i-1})$ is the volume of $B_{\mathbf{j}}$.

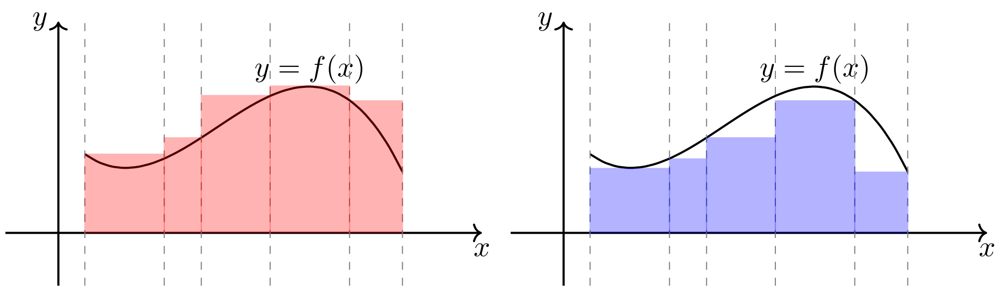

Figure 1.2: Upper (red) and lower (blue) Darboux sums of a function f over an interval ($d = 1$).

- The *upper* and *lower Darboux integral of f over B* are

$$U_B(f) = \inf\{U_B(f,P) : P \text{ is a Darboux partition of } B\}$$

and

$$L_B(f) = \sup\{L_B(f,P) : P \text{ is a Darboux partition of } B\}.$$

- The function f is *Darboux integrable over B* if $U_B(f) = L_B(f)$, and their common value is called the *Darboux integral of f over B* and is denoted by $\int_B f(\boldsymbol{x})\, d\boldsymbol{x}$.

Proposition 1.2. *A function f is Darboux integrable if and only if it is Riemann integrable. Moreover, the value of the Darboux integral and the Riemann integral (for a Riemann–Darboux integrable function) are the same.*

Definition 1.3. A bounded set $E \subseteq \mathbb{R}^d$ is a *Jordan measurable set* if $\mathbb{1}_E$ is Riemann–Darboux integrable over a box containing E. The *Jordan content* of a Jordan measurable set E is the value $J(E) = \int_B \mathbb{1}_E(\boldsymbol{x})\, d\boldsymbol{x}$, where B is any box containing E.

Jordan measurable sets include basic geometric objects such as polyhedra, conic sections, regions bounded by finitely many smooth curves/surfaces, etc.

Definition 1.4. A set $S \subseteq \mathbb{R}^d$ is a *simple set* if it is a finite union of boxes $S = \bigcup_{j=1}^k B_j$.

If the boxes $B_1, \dots, B_k$ are disjoint, then the volume of the simple set $S = \bigcup_{j=1}^k B_j$ is $\mathrm{Vol}(S) = \sum_{j=1}^k \mathrm{Vol}(B_j)$. If some of the boxes intersect, then the volume of $S = \bigcup_{j=1}^k B_j$ can be computed using inclusion-exclusion:

$$\mathrm{Vol}(S) = \sum_{j=1}^k \mathrm{Vol}(B_j) - \sum_{1 \le j_1 < j_2 \le k} \mathrm{Vol}(B_{j_1} \cap B_{j_2}) + \sum_{1 \le j_1 < j_2 < j_3 \le k} \mathrm{Vol}(B_{j_1} \cap B_{j_2} \cap B_{j_3}) - \dots$$

This expression is well-defined, since the intersection of two boxes is again a box. A Jordan measurable set is a set that is "well-approximated" by simple sets, as we will make precise now.

Definition 1.5. For a bounded set $E \subseteq \mathbb{R}^d$, define the *inner* and *outer Jordan content* by

$$J_*(E) = \sup \{\mathrm{Vol}(S) : S \subseteq E \text{ is a simple set}\}.$$

and

$$J^*(E) = \inf \{\mathrm{Vol}(S) : S \supseteq E \text{ is a simple set}\}.$$

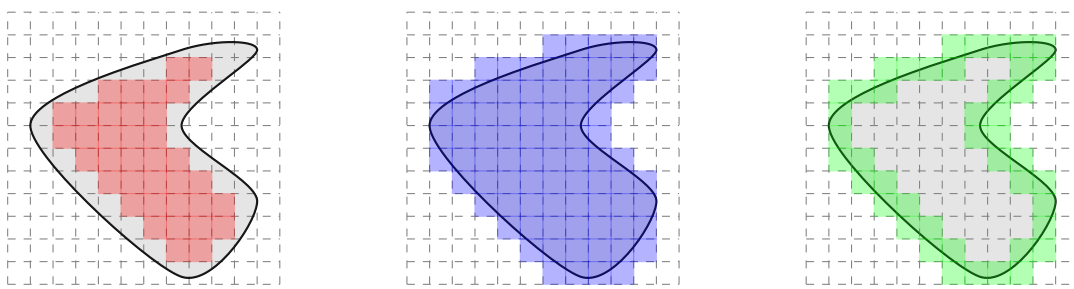

Figure 1.3: Simple sets approximating the inner (red) and outer Jordan content (blue) of a region in dimension $d = 2$. With the red boxes removed from the blue, we get a simple set covering the boundary (in green).

Theorem 1.6. *Let $E \subseteq \mathbb{R}^d$ be a bounded set. The following are equivalent:*

(i) E is Jordan measurable;

(ii) $J_*(E) = J^*(E)$ (in which case $J(E)$ is equal to this same value);

(iii) $J^*(\partial E) = 0$.

Proof. We will prove the $d = 1$ case. The multidimensional case is similar but more notationally cumbersome, so we omit it to avoid additional technical details that would largely obscure the main ideas.

(i) $\iff$ (ii). To establish this equivalence, it suffices to show

$$U_B(\mathbb{1}_E) = J^*(E) \qquad \text{and} \qquad L_B(\mathbb{1}_E) = J_*(E)$$

for any box (interval) $B \supseteq E$. Let us prove $U_B(\mathbb{1}_E) = J^*(E)$.

<u>Claim</u>: $U_B(\mathbb{1}_E) \le J^*(E)$.

Let $\varepsilon > 0$. Then from the definition of the outer Jordan content, there exists a simple set $S \subseteq \mathbb{R}$ such that $E \subseteq S$ and $\mathrm{Vol}(S) < J^*(E) + \varepsilon$. By assumption, B is an interval containing E, so $S \cap B$ is also a simple set containing E, and $\mathrm{Vol}(S \cap B) \leq \mathrm{Vol}(S) < J^*(E) + \varepsilon$. We may therefore assume without loss of generality that $S \subseteq B$. Write $B = [a, b]$ and $S = [a_1, b_1] \sqcup [a_2, b_2] \sqcup \cdots \sqcup [a_n, b_n]$ with $a \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n \leq b$. We define a Darboux partition[1] P of $[a, b]$ by $P = (x_i)_{i=0}^{2n+1}$ with $x_0 = a$, $x_1 = a_1$, $x_2 = b_1$, $\ldots$, $x_{2n-1} = a_n$, $x_{2n} = b_n$, $x_{2n+1} = b$. Then since $E \subseteq S$, we have

$$\begin{aligned} U_B(\mathbb{1}_E, P) &= \sum_{i=1}^{2n+1} \sup_{x_{i-1} \leq x \leq x_i} \mathbb{1}_E(x) \cdot (x_i - x_{i-1}) \\ &\leq 0 \cdot (a_1 - a) + 1 \cdot (b_1 - a_1) + 0 \cdot (a_2 - b_1) + \cdots + 1 \cdot (b_n - a_n) + 0 \cdot (b - b_n) \\ &= \mathrm{Vol}(S). \end{aligned}$$

Hence, $U_B(\mathbb{1}_E) \leq U_B(\mathbb{1}_E, P) \leq \mathrm{Vol}(S) < J^*(E) + \varepsilon$. This proves the claim.

Claim: $J^*(E) \leq U_B(\mathbb{1}_E)$.

Let $\varepsilon > 0$. Write $B = [a, b]$. Then there exists a Darboux partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $U_B(\mathbb{1}_E, P) < U_B(\mathbb{1}_E) + \varepsilon$. Let $M_i = \sup_{x_{i-1} \leq x \leq x_i} \mathbb{1}_E(x) \in \{0, 1\}$, and note that, by definition, $U_B(\mathbb{1}_E, P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$. Let $I \subseteq \{1, \ldots, n\}$ be the set $I = \{1 \leq i \leq n : M_i = 1\}$, and let $S = \bigcup_{i \in I}[x_{i-1}, x_i]$. Then S is a simple set with length $\mathrm{Vol}(S) = \sum_{i \in I}(x_i - x_{i-1}) = U_B(\mathbb{1}_E, P)$. Moreover, $E \subseteq S$, since S is the union of all intervals that have nonempty intersection with E. Thus, $J^*(E) \leq \mathrm{Vol}(S) = U_B(\mathbb{1}_E, P) < U_B(\mathbb{1}_E) + \varepsilon$.

The identity $L_B(\mathbb{1}_E) = J_*(E)$ is proved similarly.

(ii) $\iff$ **(iii)**. It suffices to prove $J^*(\partial E) = J^*(E) - J_*(E)$. (See Figure 1.3.)

Claim: $J^*(\partial E) \leq J^*(E) - J_*(E)$.

Let $\varepsilon > 0$. Let S_1 be a simple set such that $E \subseteq S_1$ and $\mathrm{Vol}(S_1) < J^*(E) + \frac{\varepsilon}{2}$. Since S_1 is closed, we have $\overline{E} \subseteq S_1$. Let S_2 be a simple set with $S_2 \subseteq E$ such that $\mathrm{Vol}(S_2) > J_*(E) - \frac{\varepsilon}{2}$. Note that $\mathrm{int}(S_2) \subseteq \mathrm{int}(E)$. Therefore, $S = S_2 \setminus \mathrm{int}(S_1)$ is a simple set and $\partial E = \overline{E} \setminus \mathrm{int}(E) \subseteq S$, so $J^*(\partial E) \leq \mathrm{Vol}(S) = \mathrm{Vol}(S_2) - \mathrm{Vol}(S_1) < J^*(E) - J_*(E) + \varepsilon$. But ε was arbitrary, so we conclude $J^*(\partial E) \leq J^*(E) - J_*(E)$.

Claim: $J^*(E) - J_*(E) \leq J^*(\partial E)$.

Let $\varepsilon > 0$, and let $S \supseteq \partial E$ be a simple set with $\mathrm{Vol}(S) < J^*(\partial E) + \frac{\varepsilon}{2}$. Write $S = \bigsqcup_{i=1}^n [a_i, b_i]$ with $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$. Let $[a, b] \subseteq \mathbb{R}$ such that $E \subseteq [a, b]$ and $a < a_1$ and $b < b_n$. For notational convenience, let $b_0 = a$ and $a_{n+1} = b$. Let $I \subseteq \{0, \ldots, n\}$ be the collection of indices i such that $(b_i, a_{i+1}) \cap E \neq \emptyset$. For each $i \in I$, we claim that $(b_i, a_{i+1}) \subseteq E$. If not, then (b_i, a_{i+1}) contains a boundary point of E, but $\partial E \subseteq S$, so this is a contradiction. Thus, $S' = \bigcup_{i \in I}[b_i, a_{i+1}]$ is a simple set with $\mathrm{int}(S') \subseteq E$. Shrinking slightly each interval in S', we obtain a simple set

$$S'' = \bigcup_{i \in I} \left[b_i + \frac{\varepsilon}{4(n+1)}, a_{i+1} - \frac{\varepsilon}{4(n+1)} \right]$$

[1] Strictly speaking, this may fail to be a Darboux partition, since some of the points are allowed to coincide. However, the value we compute for $U_B(\mathbb{1}_E, P)$ will be the correct value for the partition where we remove repetitions of the same point.

such that $S'' \subseteq E$. Moreover, $\mathrm{Vol}(S'') \geq \mathrm{Vol}(S') - \frac{\varepsilon}{2(n+1)}|I| \geq \mathrm{Vol}(S') - \frac{\varepsilon}{2}$. Noting that $S \cup S'$ is a simple set containing E, we arrive at the inequality

$$J^*(E) - J_*(E) \leq \mathrm{Vol}(S \cup S') - \mathrm{Vol}(S'') = \mathrm{Vol}(S) + \mathrm{Vol}(S') - \mathrm{Vol}(S'') < J^*(\partial E) + \varepsilon.$$

□

Example 1.7. The sets $\mathbb{Q} \cap [0,1]$ and $[0,1] \setminus \mathbb{Q}$ are not Jordan measurable (see Problem 1.1).

In addition to the above example, there are many other "nice" sets that are not Jordan measurable. There are, for instance, bounded open sets in $\mathbb{R}$ that are not Jordan measurable. We will work out one such example in detail.

Example 1.8. The complement U of the fat Cantor set (also known as the Smith–Volterra–Cantor set) $K \subseteq [0,1]$ is Jordan non-measurable. We construct K iteratively, starting from $[0,1]$, by removing intervals of length 4^{-n} at step n. In other words, at step n, we remove an interval of length 4^{-n} around each rational point with denominator 2^n.

Figure 1.4: Iterative construction of the fat Cantor set.

Let

$$U = \bigcup_{n=0}^{\infty} \bigcup_{j=1}^{2^n} \left(\frac{2j+1}{2^{n+1}} - \frac{1}{2 \cdot 4^{n+1}}, \frac{2j+1}{2^{n+1}} + \frac{1}{2 \cdot 4^{n+1}} \right).$$

Then $K = [0,1] \setminus U$.

The inner Jordan content of U is

$$J_*(U) = \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} \mathrm{Len} \left(\frac{2j+1}{2^{n+1}} - \frac{1}{2 \cdot 4^{n+1}}, \frac{2j+1}{2^{n+1}} + \frac{1}{2 \cdot 4^{n+1}} \right) = \sum_{n=0}^{\infty} 2^n \cdot \frac{1}{4^{n+1}} = \frac{1}{4} \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{2}.$$

However, $\overline{U} = [0,1]$ (since U contains every rational number whose denominator is a power of 2), so the outer Jordan content of U is $J^*(U) = J^*([0,1]) = 1$.

1.3 Limits of Integrable Functions

You may recall from the theory of Riemann integration that *uniform* limits of Riemann integrable functions are Riemann integrable, and one may in this case interchange the order of taking limits and computing the integral. More precisely:

Theorem 1.9. *Let B be a box in $\mathbb{R}^d$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of Riemann integrable functions on B, and suppose f_n converges uniformly to a function $f : B \to \mathbb{R}$. Then f is Riemann integrable, and*

$$\int_B f(\boldsymbol{x})\ d\boldsymbol{x} = \lim_{n \to \infty} \int_B f_n(\boldsymbol{x})\ d\boldsymbol{x}.$$

One of the deficiencies of the Riemann–Darboux–Jordan approach to integration and measurement is that pointwise (non-uniform) limits do not share this property.

Example 1.10. Enumerate the set $\mathbb{Q} \cap [0,1] = \{q_1, q_2, \dots\}$. Let $f_n : [0,1] \to [0,1]$ be the function

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{q_1, \dots, q_n\} \\ 0, & \text{otherwise.} \end{cases}$$

Then f_n is Riemann integrable and $f_n \to \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ pointwise, but $\mathbb{1}_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable.

Since analysis so often deals with limits, it is desirable to develop a theory of integration that accommodates pointwise limits. The Lebesgue measure and Lebesgue integral resolve this shortcoming.

1.4 The Solution of Lebesgue

The Jordan non-measurable set in Example 1.8 appears to have a sensible notion of "length." Indeed, the complement U, being a disjoint union of intervals, could be reasonably assigned as a "length" the sum of the lengths of the (countably many) intervals of which it is made. This produces a value of $\frac{1}{2}$ for the length of U, and so we should take K to also have length $\frac{1}{2}$, since $K \sqcup U = [0,1]$ is an interval of length 1. The feature that U is a disjoint union of intervals turns out to not be any special feature of U at all but instead a general feature of open sets in $\mathbb{R}$.

Proposition 1.11. *Let $U \subseteq \mathbb{R}$ be an open set. Then U can be expressed as a countable disjoint union of open intervals.*

Proof. Problem 1.2. □

By Proposition 1.11, it seems reasonable to define the length of an open set $U \subseteq \mathbb{R}$ as follows. Write $U = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \dots$ as a disjoint union of open intervals, and define its length as $(b_1 - a_1) + (b_2 - a_2) + \dots$. Then open sets may play the role that simple sets played in the definition of the Jordan content, and this leads to the Lebesgue measure.

Remark 1.12. In higher dimensions, Proposition 1.11 needs to be modified, but one can still reasonably talk about the d-dimensional volume of open sets in $\mathbb{R}^d$. See Problems 1.3 and 1.4.

Definition 1.13. Let $E \subseteq \mathbb{R}^d$.

- The *outer Lebesgue measure of* E is the quantity

$$\begin{aligned} \lambda^*(E) &= \inf \{\mathrm{Vol}(U) : U \supseteq E \text{ is open}\} \\ &= \inf \left\{ \sum_{j=1}^{\infty} \mathrm{Vol}(B_j) : B_1, B_2, \dots \text{ are boxes, and } E \subseteq \bigcup_{j=1}^{\infty} B_j \right\}. \end{aligned}$$

- The set E is *Lebesgue measurable* (with Lebesgue measure $\lambda(E) = \lambda^*(E)$) if for every $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^d$ such that $E \subseteq U$ and $\lambda^*(U \setminus E) < \varepsilon$.

Proposition 1.14. *If $E \subseteq \mathbb{R}^d$ is Jordan measurable, then E is Lebesgue measurable and $J(E) = \lambda(E)$.*

The family of Lebesgue measurable sets is much larger than the family of Jordan measurable sets. Among the several nice properties of the Lebesgue measure (and abstract measures) that we will see later in the course are:

Proposition 1.15.

(1) If $(E_n)_{n\in\mathbb{N}}$ are Lebesgue measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ and $\bigcap_{n=1}^{\infty} E_n$ are Lebesgue measurable.

(2) If $(E_n)_{n\in\mathbb{N}}$ are pairwise disjoint and Lebesgue measurable, then $\lambda\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n)$.

(3) If $E_1 \subseteq E_2 \subseteq \cdots \subseteq \mathbb{R}^d$ are Lebesgue measurable sets, then $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n)$.

(4) If $E_1 \supseteq E_2 \supseteq \dots$ are Lebesgue measurable subsets of $\mathbb{R}^d$ and $\lambda(E_1) < \infty$, then $\lambda\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n)$.

1.5 Applications of Abstract Measure Theory

The mathematical language and tools encompassed in measure theory play a foundational role in many other areas of mathematics. A highly abbreviated sampling follows.

Probability theory. Measure theory provides the axiomatic foundations of probability theory, providing rigorous notions of *random variables* and *probabilities* of *events*. Important limit laws (the law of large numbers and central limit theorem, for example) are phrased mathematically using measure-theoretic notions of convergence.

Fourier analysis. Periodic (say, continuous or Riemann-integrable) functions on the real line have corresponding Fourier series representations $f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$. The functions $e^{2\pi i n x}$ are orthonormal, and Parseval's identity gives $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \int_0^1 |f(x)|^2 \, dx$. Given a sequence $(a_n)_{n\in\mathbb{N}}$, one may ask whether $\sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$ is the Fourier expansion of some function f, and if so, what properties does f have? Another natural question is whether the series $\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$ actually converges to the function f, and if so, in which sense? Both of these questions are properly answered in a measure-theoretic framework. If one is interested in decomposing functions defined on other groups (for instance, on compact abelian groups) into their Fourier series, then one also needs to develop a method of integrating functions on groups in order to compute Fourier coefficients and make sense of Parseval's identity.

Functional analysis and operator theory. When one studies familiar concepts from linear algebra in infinite-dimensional spaces, measures become unavoidable for many tasks. For example, versions of the spectral theorem (generalizing the representation of suitable matrices in terms of their eigenvalues and eigenvectors) for operators on infinite-dimensional spaces require the abstract notion of a measure.

Ergodic theory. Ergodic theory was developed to study the long-term statistical behavior of dynamical (time-dependent) systems, providing a framework to resolve important problems in physics related to the "ergodic hypothesis" in thermodynamics and the "stability" of the solar system. It turns out that the appropriate mathematical formalism for understanding these problems comes from abstract measure theory.

Fractal geometry. Self-similar geometric objects such as the Koch snowflake, Sierpiński carpet, and the middle-thirds Cantor set (see Figure 1.5) can be meaningfully assigned a notion of "dimension" that can take a non-integer value. How does one determine the dimension of a fractal object? There are several different approaches to dimension, but one of the most popular is the *Hausdorff*

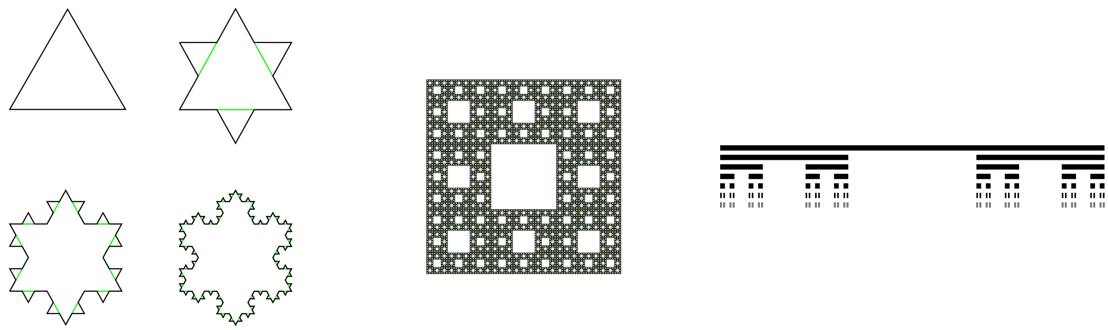

Figure 1.5: Fractal shapes: the Koch snowflake (left) of Hausdorff dimension $\frac{\log 4}{\log 3} \approx 1.26$, Sierpiński carpet (middle) of dimension $\frac{\log 8}{\log 3} \approx 1.89$, and middle-thirds Cantor set (right) of dimension $\frac{\log 2}{\log 3} \approx 0.63$.

dimension, which relies on a family of measures that interpolate between the integer-dimensional Lebesgue measures.

Additional Reading

This introductory chapter is heavily influenced by the book of Tao [1] on measure theory. Many of the results in this chapter are discussed in greater detail in [1, Section 1.1].

1.6 Exercises

Problem 1.1. Show that $J^*(\mathbb{Q} \cap [0,1]) = J^*([0,1] \setminus \mathbb{Q}) = 1$, and $J_*(\mathbb{Q} \cap [0,1]) = J_*([0,1] \setminus \mathbb{Q}) = 0$.

Problem 1.2. Let $U \subseteq \mathbb{R}$ be an open set. Show that U can be written as a disjoint union of countably many open intervals.

Problem 1.3. Let $U = \{(x,y) : x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ be the open unit disk. Show that U cannot be expressed as a disjoint union of countably many open boxes.

Problem 1.4. Let $U \subseteq \mathbb{R}^d$ be an open set. Show that U can be written as a disjoint union of countably many half-open boxes (i.e., sets of the form $B = \prod_{i=1}^d [a_i, b_i)$).

Bibliography

[1] Terence Tao. *An Introduction to Measure Theory*. Grad. Stud. Math., **126** (American Mathematical Society, Providence, RI, 2011). Preliminary version available online at `https://terrytao.wordpress.com/wp-content/uploads/2012/12/gsm-126-tao5-measure-book.pdf`